

ON THE LOCALIZATION OF HV-CONVEX COMPACT PLANAR SETS GIVEN BY THEIR COORDINATE X-RAY FUNCTIONS

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ABSTRACT. The object of the paper is the problem of localization of hv-convex compact planar sets given by their coordinate X-ray functions. The first step is to reconstruct the so-called generalized conic function (GC function) associated to a compact planar set with non-empty interior from its coordinate X-rays. This function involves the coordinate X-rays as second order partial derivatives except on a set of measure zero. The main result states that the mapping Φ corresponding GC functions to the sets is continuous with respect to the Hausdorff metric on the collection of non-empty compact and hv-convex sets having the same box. We also have that its inverse (as a set-valued mapping) is upper semi-continuous admitting an approximating process via GC functions. Examples based on a kind of greedy algorithm are also presented. The last section is devoted to the case of compact convex planar bodies. We claim that the unicity of the body with given coordinate X-rays depends on the lower-semicontinuity. Radström's embedding theorem admits to formulate the problem of unicity due to R. J. Gardner in terms of set-valued analysis: how to provide the lower-semicontinuity of the inverse of a bounded order-preserving concave and continuous mapping $\Phi: C \subset V \rightarrow W$, where C is a convex compact subset, V and W are Kakutani vector spaces.

1. PRELIMINARIES

The idea motivating our investigations is the application of generalized conics' theory [6], [7] and [8] in geometric tomography. Let K be a compact planar set with non-empty interior in the euclidean plane \mathbb{R}^2 , $p \geq 1$ and consider the distance function

$$d_p((x, y), (\alpha, \beta)) = \sqrt[p]{|x - \alpha|^p + |y - \beta|^p}$$

induced by the p -norm. It can be easily seen that

$$\sqrt{2}d_2((x, y), (\alpha, \beta)) \geq d_1((x, y), (\alpha, \beta)) \geq d_2((x, y), (\alpha, \beta)).$$

Pairs of the form (x, y) , (α, β) and (c_1, c_2) denote elements of \mathbb{R}^2 . Let us define the sets

$$\begin{aligned} x < K &:= \{(\alpha, \beta) \in K \mid x < \alpha\}, & K < x &:= \{(\alpha, \beta) \in K \mid \alpha < x\}, \\ y < K &:= \{(\alpha, \beta) \in K \mid y < \beta\}, & K < y &:= \{(\alpha, \beta) \in K \mid \beta < y\}, \\ x = K &:= \{(\alpha, \beta) \in K \mid \alpha = x\}, & y = K &:= \{(\alpha, \beta) \in K \mid \beta = y\}. \end{aligned}$$

1991 *Mathematics Subject Classification.* 26B15, 26B25.

Key words and phrases. Hausdorff metric, parallel X-rays, set-valued mappings.

Definition 1. *The X-ray functions into the coordinate directions are*

$$Y(x) := L(x = K) \quad (\text{the measure of the set } x = K),$$

$$X(y) := L(y = K) \quad (\text{the measure of the set } y = K).$$

X-ray functions (especially coordinate X-rays) are typical objects in geometric tomography [2]. We start from a compact set with non-empty interior in the plane to construct a convex function carrying the information of coordinate X-rays as second order derivatives.

Definition 2. [8] *The generalized conic function F_K associated to K is defined by the formula*

$$F_K(x, y) := \frac{1}{A(K)} f_K(x, y),$$

where $A(K)$ is the area (with respect to the Lebesgue measure) of K and

$$f_K(x, y) := \int_K (\alpha, \beta) \mapsto d_1((x, y), (\alpha, \beta)).$$

The levels of the function F_K (or, equivalently f_K) are called generalized conics with K as the set of foci.

Remark 1. Since K has a non-empty interior $A(K) > 0$ and the definition of F_K is correct. In case of the not weighted function f_K it is enough to require that K is a non-empty compact set.

The generalized conic function f_K and its weighted version F_K measure the average "taxicab" distance of the points from K via integration. Both of these functions are convex (independently of the convexity of K) and they satisfy the growth conditions

$$\liminf_{r \rightarrow \infty} \frac{F_K(x, y)}{r} > 0, \quad \text{where } r := \sqrt{x^2 + y^2},$$

$$\liminf_{r \rightarrow \infty} \frac{f_K(x, y)}{r} > 0, \quad \text{where } r := \sqrt{x^2 + y^2}$$

provided that $A(K) > 0$. Therefore the levels of the function F_K (or, equivalently f_K) are compact convex subsets in the coordinate plane. The main motivating result in [7] is that the functions F_K and f_K can be explicitly expressed in terms of the coordinate X-ray functions and the second order partial derivatives coincide the coordinate X-rays up to a constant proportional term almost everywhere. We have,

$$D_1 F_K(x, y) = \frac{A(K < x)}{A(K)} - \frac{A(x < K)}{A(K)},$$

$$D_2 F_K(x, y) = \frac{A(K < y)}{A(K)} - \frac{A(y < K)}{A(K)},$$

where, by the Cavalieri's principle,

$$A(K < x) = \int_{-\infty}^x Y(s) ds, \quad A(x < K) = \int_x^{\infty} Y(s) ds \quad \text{and} \quad A(K) = \int_{-\infty}^{\infty} Y(s) ds,$$

$$A(K < y) = \int_{-\infty}^y X(t) dt, \quad A(y < K) = \int_y^{\infty} X(t) dt \quad \text{and} \quad A(K) = \int_{-\infty}^{\infty} X(t) dt.$$

By Lebesgue differentiation theorem

$$D_1 D_1 F_K(x, y) = \frac{2}{A(K)} Y(x) \quad \text{and} \quad D_2 D_2 F_K(x, y) = \frac{2}{A(K)} X(y)$$

hold almost everywhere showing that $f_K = f_L$ if and only if K and L have the same coordinate X-rays almost everywhere and $F_K = F_L$ if and only if the coordinate X-rays of the sets are proportional to each other almost everywhere. Generalized conic functions can be considered as an accumulation of coordinate X-rays' information. The problem we want to discuss here can be formulated as follows:

- if K is a non-empty hv-convex compact planar set given by its coordinate X-ray functions find an approximating set L which is relatively close (in some sense) to at least one of the sets with the same coordinate X-rays as K almost everywhere.

The solution of the problem is divided into the following steps.

1st step. From the given coordinate X-ray functions we reconstruct the generalized conic function f_K associated to a fictive set K .

Definition 3. The outer parallel body K^ε is the union of closed Euclidean balls centered at the points of K with radius $\varepsilon > 0$.

Definition 4. [4] The Hausdorff distance between two non-empty compact sets K and L is given by the formula

$$H(K, L) := \min\{\varepsilon > 0 \mid K \subset L^\varepsilon \text{ and } L \subset K^\varepsilon\}.$$

2nd step. Taking the metric space $\mathcal{F}(\mathbb{R}^2)$ of nonempty bounded and closed (i.e. compact) subsets in the plane equipped with the Hausdorff metric we have to find a compact set $\mathcal{C}^+ \subset \mathcal{F}(\mathbb{R}^2)$ such that $f_K \in \Phi(\mathcal{C}^+)$ and the mapping $\Phi: L \in \mathcal{C}^+ \subset \mathcal{F}(\mathbb{R}^2) \rightarrow \Phi(L) := f_L$ is continuous.

3rd step. Consider the metric space $\mathcal{F}(\mathcal{F}(\mathbb{R}^2))$ of nonempty bounded and closed subsets in $\mathcal{F}(\mathbb{R}^2)$ equipped with the Hausdorff metric. By the continuity of the mapping Φ for all $\varepsilon > 0$ there exists $\delta > 0$ depending on f_K and ε such that $|f_L - f_K| < \delta$ implies that

$$L \in (\Phi^{-1}(f_K)^\varepsilon \cap \mathcal{C}^+.$$

Therefore if $f_L \approx f_K$ then L approximates at least one of the sets in \mathcal{C}^+ with the same coordinate X-rays as K almost everywhere.

In the course of these steps we have to use increasingly the qualities of the set K . The best one can require is the convexity but the aim of the paper is to formulate results in the more general situations of

1. nonempty compact sets (the upper semi-continuity of the function Φ , see Lemma 1),
2. nonempty compact sets with the same box (box-technic, localization of the convex hull, see Theorem 2),
3. nonempty compact hv-convex sets with the same box (the continuity of the function Φ , localization of the set, see Theorem 5).

The philosophy is to prefer perturbations (such as the operator of the convex hull) instead of the conceptual specification of the set K . The more general results can be easily applied to the case of compact convex planar bodies together with all of advantages of the convexity. Among others the X-ray functions are continuous on their supporting intervals and the term *almost everywhere* can be omitted. But sets are not uniquely determined by their coordinate X-rays in general even if they are convex and compact. The problem of characterizing those compact convex planar bodies that can be determined by the coordinate X-rays is due to R. J. Gardner [2]. We claim that the unicity of the body with given coordinate X-rays depends on a further regularity property (esp. lower-semicontinuity) of the inverse mapping given by reversing the correspondence of GC functions to (compact convex planar) bodies. Using Radström's embedding theorem for non-empty compact convex sets in the plane we can formulate the problem in terms of the abstract language of set-valued analysis: under what condition we can provide the continuity of the inverse (as a set-valued mapping) of a bounded order-preserving concave and continuous mapping $\Phi: C \subset V \rightarrow W$, where C is a convex compact subset, V and W are Kakutani vector spaces.

2. GENERAL OBSERVATIONS

Lemma 1. *Let L_n be a sequence of non-empty compact sets in the plane. If $L_n \rightarrow L$ with respect to the Hausdorff metric then*

$$(1) \quad \limsup_{n \rightarrow \infty} f_{L_n}(x, y) \leq f_L(x, y) \quad ((x, y) \in \mathbb{R}^2),$$

i.e. the mapping $\Phi: L \mapsto f_L$ is upper semi-continuous.

Proof. Since Φ preserves the ordering with respect to the inclusion we have that

$$f_{L_n}(x, y) \leq f_{L^{r_n}}(x, y) = f_L(x, y) + \int_{L^{r_n} \setminus L} (\alpha, \beta) \mapsto d_1((x, y), (\alpha, \beta)),$$

where $r_n := H(L_n, L)$. The integrand is obviously bounded. Therefore

$$(2) \quad f_{L_n}(x, y) \leq f_L(x, y) + M(x, y, n)(A(L^{r_n}) - A(L)),$$

where $\limsup_{n \rightarrow \infty} M(x, y, n) < \infty$. Thus we have that

$$\limsup_{n \rightarrow \infty} f_{L_n}(x, y) \leq f_L(x, y)$$

because the area of the outer parallel bodies tends to the area of L by the continuity of the Lebesgue measure from above. \square

I. Let K be a non-empty compact planar set and consider the upper level set $U_K := \{ L \in \mathcal{F}(\mathbb{R}^2) \mid f_L \geq f_K \}$ under the mapping Φ . The relation $f_L \geq f_K$ means that

$$f_L(x, y) \geq f_K(x, y) \quad \text{for all } (x, y) \in \mathbb{R}^2.$$

Lemma 2. *For any non-empty compact planar set K , the upper level set U_K is closed with respect to the Hausdorff metric and the mapping*

$$\Phi: L \in U_K \rightarrow \Phi(L) := f_L$$

is continuous at the points of $\Phi^{-1}(f_K)$, i.e. the Hausdorff convergence of the sets implies the pointwise convergence of the conic functions.

Proof. Suppose that the sequence $L_n \in U_K$ tends to L in the sense of the Hausdorff metric. Since $f_{L_n} \geq f_K$

$$f_K \leq \liminf_{n \rightarrow \infty} f_{L_n}(x, y) \leq \limsup_{n \rightarrow \infty} f_{L_n}(x, y) \leq f_L(x, y)$$

because of Lemma 1. This means that U_K is closed. If $L \in \Phi^{-1}(f_K)$ then we can write

$$f_K(x, y) \leq \liminf_{n \rightarrow \infty} f_{L_n}(x, y) \leq \limsup_{n \rightarrow \infty} f_{L_n}(x, y) \leq f_L(x, y) = f_K(x, y)$$

and, consequently,

$$\liminf_{n \rightarrow \infty} f_{L_n}(x, y) = \limsup_{n \rightarrow \infty} f_{L_n}(x, y) = f_K(x, y)$$

as was to be proved. \square

II. The box technic. Let K be a non-empty compact planar set and denote B the intersection of rectangles having parallel sides to the coordinate directions and containing K .

Definition 5. We say that B is a box for K if for all interior points $(x, y) \in B$ none of the sets $x = K$ and $y = K$ are empty.

Remark 2. If B is non-degenerate then we can use affine transformations to provide that B is a square without loss of generality.

Remark 3. It can be easily seen that if the zeros of the coordinate X-rays in the open intervalls $]a, b[$ and $]c, d[$ are isolated then none of the sets $x = K$ and $y = K$ are empty for all interior points $(x, y) \in B := [a, b] \times [c, d]$. Especially the box of compact **convex** bodies¹ are the products of the support intervals of the coordinate X-rays.

Lemma 3. The set of non-empty compact sets having boxes is closed with respect to the Hausdorff metric. The box mapping $L \mapsto B(L)$ is continuous on its domain (consisting of non-empty compact sets having boxes).

Proof. Consider the operator of the convex hull $\text{conv} : L \rightarrow \text{conv } L$. If $K \subset L^\varepsilon$ then

$$\text{conv } K \subset \text{conv } (L^\varepsilon) \subset (\text{conv } L)^\varepsilon$$

because $L^\varepsilon \subset (\text{conv } L)^\varepsilon$ which is a convex set. Therefore the operator of the convex hull is Lipschitz continuous, i.e.

$$H(\text{conv } K, \text{conv } L) \leq H(K, L).$$

Especially $\text{conv } L_n \rightarrow \text{conv } L$ if L_n tends to L with respect to the Hausdorff metric. Since the box of a compact convex set is the product of its orthogonal projections onto the coordinate axes it can be easily seen that

$$B(L_n) = B(\text{conv } L_n) \rightarrow B(\text{conv } L).$$

To finish the proof we claim that $B(\text{conv } L)$ is the box of L . Suppose, in contrary that for some interior point $(x_*, y_*) \in B(\text{conv } L)$ the set $L = x_*$ is empty. This means that L can be divided into the union of two nonempty compact sets L_1 and L_2 separated by the vertical line $x = x_*$. According to the Hausdorff convergence $L_n \rightarrow L$ the same must be true for L_n except at

¹A nonempty compact set is called a body if it is the closure of its interior.

most finite many indeces which is a contradiction because L_n has a box and $(x_*, y_*) \in B(L_n)$ is an interior point if n is great enough. Therefore

$$B(L_n) = B(\text{conv } L_n) \rightarrow B(\text{conv } L) = B(L)$$

as was to be proved. \square

Lemma 4. *Suppose that K has a non-degenerate box B and let \mathcal{C} be the collection of non-empty compact sets having the same box as K . Then \mathcal{C} is compact with respect to the Hausdorff metric and it is convex in the sense that $L_1 \in \mathcal{C}$ and $L_2 \in \mathcal{C}$ implies that $tL_1 + (1-t)L_2 \in \mathcal{C}$ for all $0 \leq t \leq 1$.*

Proof. Since \mathbb{R}^2 is complete, the space $\mathcal{F}(\mathbb{R}^2)$ equipped with the Hausdorff metric is also complete, see [1]. By Hausdorff's theorem any closed and totally bounded subset in a complete metric space is compact. According to Lemma 3, the set \mathcal{C} is closed (and, consequently, it is complete as a metric space). On the other hand for any $\varepsilon > 0$ consider the partition of the box into subrectangles B_1, \dots, B_r with diameter less than ε . Taking a set $L \in \mathcal{C}$ let us define C_L as the union of B_i 's such that $B_i \cap L \neq \emptyset$. It is clear that $B(L) = B(C_L) = B$ and we have finite many sets C_1, \dots, C_s as L runs through the set \mathcal{C} . According to the constructing process for all $L \in \mathcal{C}$ there is a set C_i such that $C_L = C_i$ and, consequently, $H(L, C_i) < \varepsilon$. This means that the collection of open balls centered at C_i 's with common radius ε is a finite covering, i.e. \mathcal{C} is totally bounded. The convexity of \mathcal{C} is clear because of the definition of a box. \square

Theorem 1. *Suppose that K has a non-degenerate box B and let \mathcal{C} be the collection of non-empty compact sets having the same box as K . The mapping $\Phi: L \in \mathcal{C} \rightarrow \Phi(L) := f_L$ is bounded and concave in the sense that*

$$\Phi(tL_1 + (1-t)L_2)(x, y) \geq t\Phi(L_1)(x, y) + (1-t)\Phi(L_2)(x, y)$$

for all points $(x, y) \in \mathbb{R}^2$ and $0 \leq t \leq 1$.

Proof. Since the mapping Φ preserves the ordering with respect to the inclusion it follows that $\Phi(L)(x, y) \leq \Phi(B)(x, y)$ for all $L \in \mathcal{C}$ and, consequently, Φ is bounded with respect to the norm

$$|f_L| := \sup_{(x, y) \in B} |f_L(x, y)|.$$

On the other hand if $(\alpha, \beta) \in B$ is an interior point, then we have

$$(tL_1 + (1-t)L_2 = \alpha) \supset (tL_1 = t\alpha) + ((1-t)L_2 = (1-t)\alpha),$$

$$(tL_1 + (1-t)L_2 = \beta) \supset (tL_1 = t\beta) + ((1-t)L_2 = (1-t)\beta).$$

Let

$$c_1 := \inf\{y \mid (t\alpha, y) \in tL_1\} = t \inf\{y \mid (\alpha, y) \in L_1\},$$

$$d_1 := \sup\{y \mid (t\alpha, y) \in tL_1\} = t \sup\{y \mid (\alpha, y) \in L_1\}$$

and

$$c_2 := \inf\{y \mid ((1-t)\alpha, y) \in (1-t)L_2\} = (1-t) \inf\{y \mid (\alpha, y) \in L_2\},$$

$$d_2 := \sup\{y \mid ((1-t)\alpha, y) \in (1-t)L_2\} = (1-t) \sup\{y \mid (\alpha, y) \in L_2\}.$$

These are well-defined numbers because none of the sets $L_i = \alpha$ ($i = 1, 2$) are empty. Then

$$P := (t\alpha, c_1) + (1-t)(L_2 = \alpha) \subset \text{conv} \{(\alpha, c_1 + c_2), (\alpha, c_1 + d_2)\}$$

and

$$Q := ((1-t)\alpha, d_2) + t(L_1 = \alpha) \subset \text{conv} \{(\alpha, d_2 + c_1), (\alpha, d_2 + d_1)\}$$

are disjoint (except on a set of measure zero). Since

$$P \cup Q \subset (tL_1 + (1-t)L_2 = \alpha)$$

it follows that

$$(3) \quad Y_{t \cdot L_1 + (1-t) \cdot L_2}(\alpha) \geq tY_{L_1}(\alpha) + (1-t)Y_{L_2}(\alpha)$$

and, in a similar way,

$$(4) \quad X_{t \cdot L_1 + (1-t) \cdot L_2}(\beta) \geq tX_{L_1}(\beta) + (1-t)X_{L_2}(\beta).$$

Inequalities (3) and (4) imply that

$$(5) \quad f_{t \cdot L_1 + (1-t) \cdot L_2}(x, y) \geq tf_{L_1}(x, y) + (1-t)f_{L_2}(x, y) \quad (0 \leq t \leq 1)$$

for all $(x, y) \in \mathbb{R}^2$, i.e. Φ is a concave function. \square

Corollary 1. *Suppose that K has a non-degenerate box B and let \mathcal{C} be the collection of non-empty compact sets having the same box as K . For any $T \subset \mathbb{R}^2$ and $L \in \mathcal{C}$, the upper level set*

$$U_L := \{ M \in \mathcal{C} \mid f_M(x, y) \geq f_L(x, y) \text{ for all } (x, y) \in T \}$$

is convex.

Remark 4. It can be easily seen that U_L is actually the upper level set with respect to the function $\Phi: L \in \mathcal{C} \rightarrow \Phi(L) := (f_L)|_T$. In what follows $T := B$ as one of natural versions. On the other hand if $f_L(x, y) = f_B(x, y)$ at a single point (x, y) then $f_L = f_B$. Therefore we can use the notation $f_L < f_B$ without any confusion.

Theorem 2. (A theorem of localization I.) *Suppose that K has a non-degenerate box B and let \mathcal{C} be the collection of non-empty compact sets having the same box as K . If $L \in U_K$ such that $f_L < f_B$ then there exists $L_* \in U_L$ such that*

$$H(\text{conv } K, L_*) \leq \frac{\text{diam } B}{2} \min\{\delta_1, \delta_2\},$$

where

$$\delta_1 := \frac{1}{\kappa(K)} \cdot \sup_{(x,y) \in B} \frac{f_L - f_K}{f_B - f_K}, \quad \delta_2 := \sup_{(x,y) \in B} \frac{f_L - f_K}{f_B - f_L}$$

and

$$\kappa(K) := \inf_{(x,y) \in B} \frac{f_B - f_{\text{conv } K}}{f_B - f_K}(x, y) \leq 1.$$

Proof. Let $(x, y) \in \mathbb{R}^2$ be given and consider the function

$$f(t) := f_{t \cdot \text{conv } K + (1-t) \cdot B}(x, y) \quad (0 \leq t \leq 1).$$

1. We are going to prove that f is concave and continuous on its domain. In case of any real numbers $0 \leq t_1 < t_2 \leq 1$ and $0 \leq t \leq 1$ we have

$$(1 - (tt_1 + (1-t)t_2))B = (t(1-t_1) + (1-t)(1-t_2))B = \\ t(1-t_1)B + (1-t)(1-t_2)B$$

because B is a convex set². In a similar way

$$(tt_1 + (1-t)t_2) \text{conv } K = tt_1 \text{conv } K + (1-t)t_2 \text{conv } K.$$

Therefore

$$f(tt_1 + (1-t)t_2) = f_{t \cdot L_1 + (1-t) \cdot L_2},$$

where

$$L_1 := t_1 \text{conv } K + (1-t_1)B \quad \text{and} \quad L_2 := t_2 \text{conv } K + (1-t_2)B.$$

Theorem 1 shows that

$$f(tt_1 + (1-t)t_2) \geq tf_{L_1} + (1-t)f_{L_2} = tf(t_1) + (1-t)f(t_2),$$

i.e. f is concave bounded (together with Φ) and, consequently, it is continuous at any inner point of its domain. Since Radström's theorem [4] states that the collection of non-empty bounded, closed convex subsets equipped with the Hausdorff metric can be isometrically embedded into a Banach space, the metric structure of a segment is the same as that in a normed linear space, i.e.

$$(6) \quad H(t_1 L_1 + (1-t_1)L_2, t_2 L_1 + (1-t_2)L_2) = (t_2 - t_1)H(L_1, L_2)$$

provided that $0 \leq t_1 < t_2 \leq 1$. Therefore the convergence $t_n \rightarrow t$ means the Hausdorff convergence

$$t_n L_1 + (1-t_n)L_2 \rightarrow t L_1 + (1-t)L_2$$

of the corresponding convex sets. We have that f is upper semi-continuous at the endpoints (together with Φ) and the upper semi-continuity of a concave function on a bounded closed interval implies the continuity at the endpoints too.

2. Let $L \in U_K$ be a given set and consider the function

$$g(t) := \inf_{(x,y) \in B} (f_{t \cdot \text{conv } K + (1-t) \cdot B}(x, y) - f_L(x, y)) \quad (0 \leq t \leq 1).$$

According to the first part of the proof g is concave and continuous. If $f_L < f_B$ then $0 < g(0)$. If $g(1) \geq 0$ then $\text{conv } K \in U_L$ and the proof is finished by the choice $\text{conv } K = L_*$. If $g(1) < 0$ we have a number $0 < t_* < 1$ such that $g(t_*) = 0$ and, consequently, the set

$$L_* := t_* \text{conv } K + (1-t_*)B$$

is an element of the upper level set U_L such that $f_{L_*}(x_*, y_*) = f_L(x_*, y_*)$ for some $(x_*, y_*) \in B$ where the infimum is attained at. The concavity of the function

$$f_*(t) := f_{t \cdot \text{conv } K + (1-t) \cdot B}(x_*, y_*) \quad (0 \leq t \leq 1)$$

²The equation $(\lambda + \mu)M = \lambda M + \mu M$ holds if M is convex and the numbers λ and μ have a common sign but not in general, see [1].

associated to the point (x_*, y_*) implies that

$$(7) \quad \frac{f_*(t_*) - f_*(0)}{t_*} \geq \frac{f_*(1) - f_*(t_*)}{1 - t_*}$$

and

$$(8) \quad \frac{f_*(1) - f_*(0)}{1} \geq \frac{f_*(1) - f_*(t_*)}{1 - t_*}$$

which means that

$$0 > \frac{f_{L_*}(x_*, y_*) - f_B(x_*, y_*)}{t_*} \geq \frac{f_{\text{conv } K}(x_*, y_*) - f_{L_*}(x_*, y_*)}{1 - t_*}$$

and

$$0 > \frac{f_{\text{conv } K}(x_*, y_*) - f_B(x_*, y_*)}{1} \geq \frac{f_{\text{conv } K}(x_*, y_*) - f_{L_*}(x_*, y_*)}{1 - t_*}.$$

Equation (6) shows that the parameters can be written in terms of simple ratios

$$t_* = H(B, L_*) : H(\text{conv } K, B)$$

and

$$1 - t_* = H(\text{conv } K, L_*) : H(\text{conv } K, B)$$

of Hausdorff distances. On the other hand $f_L(x_*, y_*) = f_{L_*}(x_*, y_*)$ and $f_{\text{conv } K} \geq f_K$ because of $K \subset \text{conv } K$. Therefore (changing the sign of the corresponding sides) inequality (7) says that

$$H(\text{conv } K, L_*) \leq \frac{f_L(x_*, y_*) - f_{\text{conv } K}(x_*, y_*)}{f_B(x_*, y_*) - f_L(x_*, y_*)} H(B, L_*) \leq$$

$$\frac{f_L(x_*, y_*) - f_K(x_*, y_*)}{f_B(x_*, y_*) - f_L(x_*, y_*)} H(B, L_*) \leq H(B, L_*) \cdot \delta_2 \leq H(B, \text{conv } K) \cdot \delta_2.$$

From inequality (8)

$$H(\text{conv } K, L_*) \leq \frac{f_L(x_*, y_*) - f_{\text{conv } K}(x_*, y_*)}{f_B(x_*, y_*) - f_{\text{conv } K}(x_*, y_*)} H(B, \text{conv } K) \leq \frac{f_L(x_*, y_*) - f_K(x_*, y_*)}{f_B(x_*, y_*) - f_{\text{conv } K}(x_*, y_*)} H(B, \text{conv } K),$$

where

$$\frac{f_B(x_*, y_*) - f_{\text{conv } K}(x_*, y_*)}{f_B(x_*, y_*) - f_K(x_*, y_*)} \geq \kappa(K)$$

and thus

$$H(\text{conv } K, L_*) \leq H(B, \text{conv } K) \cdot \delta_1.$$

Since B is the box of $\text{conv } K$ their Hausdorff distance is at most the half of the diagonal. \square

Remark 5. Theorem 2 prefers the perturbation $\text{conv } K$ instead of the conceptual specification of the object set K .

The bounds δ_1 and δ_2 have an affine character as estimations of the simple ratio

$$H(\text{conv } K, L_*) : H(B, \text{conv } K).$$

Since δ_2 depends only on the function f_K ,

$$\text{conv } K' \subset (U_L)^\delta \quad \text{with} \quad \delta := \frac{\text{diam } B}{2} \cdot \delta_2$$

holds for all $K' \in \Phi^{-1}(f_K)$. If K is convex then $\kappa(K) = 1$ and $\delta_1 \leq \delta_2$. In case of non-convex sets the bound δ_1 is theoretical unless having an *a priori* number $\kappa(K)$ involving the difference between K and $\text{conv } K$ in terms of generalized conic functions. This number admits to majorize the unknown function $f_{\text{conv } K}$ by the help of the inequalities

$$f_{\text{conv } K}(x, y) \leq \kappa(K)f_K(x, y) + (1 - \kappa(K))f_B(x, y).$$

The Hausdorff distance between K and $\text{conv } K$ can be estimated as

$$H(K, \text{conv } K) \leq H(\text{ext conv } K, \text{conv } K),$$

where $\text{ext conv } K \subset K$ is the set of extremal points of the convex hull. This is the best in the sense of Krein-Milmann's theorem which states that for any compact convex set in the euclidean space \mathbb{R}^n is the convex hull of its extremal points. In case of $M := \text{ext conv } K$ we have that

$$\text{ext conv } M = M$$

and thus $H(M, \text{conv } M) = H(\text{ext conv } M, \text{conv } M)$.

3. THE CASE OF HV-CONVEX SETS

In what follows $K \subset \mathbb{R}^2$ is a non-empty bounded closed (i.e. compact) and hv-convex set with a non-degenerate box $B := [a, b] \times [c, d]$. Let

1. \mathcal{C} be the collection of non-empty bounded closed (i.e. compact) sets having the same box as K and denote by

$$U_K := \{ L \in \mathcal{C} \mid f_L(x, y) \geq f_K(x, y) \text{ for all } (x, y) \in B \}$$

the upper level set with respect to the function

$$\Phi: L \in \mathcal{C} \rightarrow \Phi(L) := (f_L)|_B,$$

2. \mathcal{C}^+ be the collection of non-empty bounded closed (i.e. compact) and hv-convex sets having the same box as K and denote by U_K^+ the upper level set with respect to the restricted function $\Phi|_{\mathcal{C}^+}$. It is clear that

$$U_K^+ = U_K \cap \mathcal{C}^+.$$

Lemma 5. *For any set $M \in \mathcal{C}$ the functions $f: [a, b] \rightarrow \mathbb{R}$ and $g: [a, b] \rightarrow \mathbb{R}$ defined by the formulas*

$$f(x) := \sup \{ y \mid (x, y) \in M \} \quad \text{and} \quad g(x) := \inf \{ y \mid (x, y) \in M \}$$

are upper and lower semi-continuous, respectively (similar statements can be formulated in terms of the orthogonal coordinate direction).

Proof. First of all note that f and g are well-defined on the whole interval because B is the box of the set M . Let $x \in [a, b]$ be an arbitrary point of the domain and consider a sequence $x_n \rightarrow x$. Since M is compact and $(x_n, f(x_n)) \in M$ we have that $y_* \leq f(x)$ for any convergent subsequence from $f(x_n)$ with limit y_* . Therefore $\limsup_{n \rightarrow \infty} f(x_n) \leq f(x)$ as was to be stated. The case of the „lower bound” function g is similar \square

Corollary 2. [2] *For any set $M \in \mathcal{C}$ the coordinate X-ray functions are upper semicontinuous on the interval $[a, b]$ and $[c, d]$, respectively.*

Proof. Since the coordinate X-ray function Y_M can be estimated by

$$Y_K(x) \leq f(x) - g(x)$$

for any sequence $x_n \rightarrow x$

$$\limsup_{n \rightarrow \infty} Y_K(x_n) \leq \limsup_{n \rightarrow \infty} f(x_n) - \liminf_{n \rightarrow \infty} g(x_n) \leq f(x) - g(x) = Y_K(x)$$

as was to be proved. The case of the coordinate X-ray X_K is similar. \square

Lemma 6. *For any set $M \in \mathcal{C}^+$ the outer parallel body M^ε is connected and hv-convex.*

Proof. To prove that M^ε is hv-convex suppose, in contrary, that it is not true. Then (using translations and similarities³ if necessary) we have (for example) the points $(0, 0)$ and $(0, 1)$ in M^ε such that $(0, y) \notin M^\varepsilon$, where $0 < y < 1$. According to the definition of the parallel body there exist points (x_1, y_1) and $(x_2, y_2) \in M$ in the closed disks centered at $(0, 0)$ and $(0, 1)$ with radius ε , respectively, but M must be disjoint from the closed disk D centered at the point $(0, y)$ with radius ε . Therefore $x_1 \neq x_2$ (because in opposite case the vertical line segment joining them intersects D) and, using reflection at the vertical coordinate line if necessary, we can suppose that $x_1 < x_2$ without loss of generality. By the help of the „upper bound” function

$$f: [a, b] \rightarrow \mathbb{R}, \quad f(x) := \sup \{y \mid (x, y) \in M\}$$

we have $f(x_1) < y$ because in opposite case the vertical line segment joining (x_1, y_1) and $(x_1, f(x_1))$ intersects D . Therefore

$$y_1 \leq f(x_1) < y < y_2 \leq f(x_2).$$

Let us define the number

$$0 \leq s := \sup \{t \geq 0 \mid f(x) < y \text{ for all } x \in [x_1, x_1 + t]\} \leq x_2 - x_1.$$

Then we can choose a sequence $s_n \rightarrow s^+$ such that $f(s_n) \geq y$ and, by the upper semi-continuity of the function f , it follows that

$$y \leq \limsup_{n \rightarrow \infty} f(s_n) \leq f(s).$$

On the other hand M is compact which implies that for any sequence $s_n \rightarrow s^-$ we have a limit $(s, y_*) \in M$ in case of some convergent subsequence of $(s_n, f(s_n))$. Then, of course, $y_* \leq y$ and the convexity into the

³The equation $\lambda(M + L) = \lambda M + \lambda L$ holds without any restriction for the sets and M^ε is just the sum of the set M with the closed disk centered at the origin with radius ε .

vertical direction gives a contradiction because the (vertical) segment joining $(s, f(s))$ with (s, y_*) in M intersects the closed disk D centered at the point $(0, y)$ with radius ε .

For the connectedness note that M^ε has obviously the same box as the parallel body B^ε and consider the connected component in M^ε containing $(x, y) \in M^\varepsilon$. If $[a_{(x,y)}, b_{(x,y)}]$ is the orthogonal projection of the corresponding component onto the horizontal coordinate line then, by the convexity into the vertical direction, we have that any two of such kind of intervals must be the same or disjoint. Moreover $\text{diam } [a_{(x,y)}, b_{(x,y)}] \geq 2\varepsilon$ because M^ε is the union of closed disks centered at the points of M with radius ε . Since M^ε has a box with horizontal side $[a, b]$ it follows that

$$[a, b] = \bigcup_{(x,y) \in M^\varepsilon} [a_{(x,y)}, b_{(x,y)}],$$

where the different members of the union are (pairwise) disjoint closed intervals all of whose measure is at least 2ε . Therefore M^ε must have finitely many components with pairwise disjoint projections $[a_1, b_1], \dots, [a_m, b_m]$ onto the horizontal coordinate line. According to the box-property again

$$[a, b] = \bigcup_{i=1}^m [a_i, b_i]$$

and we have $m = 1$ to avoid the contradiction. \square

Lemma 7. \mathcal{C}^+ is a compact set with respect to the Hausdorff metric.

Proof. According to Lemma 4 it is enough to prove that $\mathcal{C}^+ \subset \mathcal{C}$ is closed. Consider a sequence $L_n \in \mathcal{C}^+$ tending to the limit $L \in \mathcal{C}$ (see Lemma 3). We are going to discuss the convexity only into the horizontal direction (the discussion of the vertical convexity is similar). Suppose, in contrary, that there exist points $P_1(x_1, y), P_2(x_2, y) \in L$ such that the segment joining P_1 with P_2 contains a point $Q(x, y) \notin L$. Since L is compact $d_2(L, Q) > 0$. Let

$$0 < \varepsilon < \frac{d_2(L, Q)}{3}$$

be an arbitrary positive real number. Then $L^{2\varepsilon}$ is disjoint from the closed disk centered at Q with radius ε . The Hausdorff convergence $L_n \rightarrow L$ implies the existence of $n \in \mathbb{N}$ such that $L_n \subset L^\varepsilon$ and $L \subset L_n^\varepsilon$. Therefore there exist points $R_1, R_2 \in L_n$ in the closed disks centered at P_1 and P_2 with radius ε , respectively. Since $L_n^\varepsilon \subset (L^\varepsilon)^\varepsilon \subset L^{2\varepsilon}$ we have that L_n^ε is disjoint from the closed disk centered at Q with radius ε . Consider the (common) tangent lines e and f of the disks parallel to the horizontal direction. They are tangent to the disk centered at Q with radius ε too. Let H_e be the closed half plane bounded by e containing the line f and, in a similar way, H_f denotes the closed half plane bounded by f containing the line e . In view of Lemma 6, the parallel body L_n^ε is connected together with its interior containing L_n . Therefore $\text{int } L_n^\varepsilon$ is arcwise connected as a connected open subset of the euclidean plane. There exists a continuous arc $s \subset \text{int } L_n^\varepsilon$ joining R_1 and R_2 but $L_n^\varepsilon \subset (L^\varepsilon)^\varepsilon \subset L^{2\varepsilon}$ implies that L_n^ε together with s is also disjoint from the closed disk centered at Q with radius ε . Taking a point $S \in s$ such that $S \notin H_e \cap H_f$ (suppose, for example, that $S \notin H_e$) we

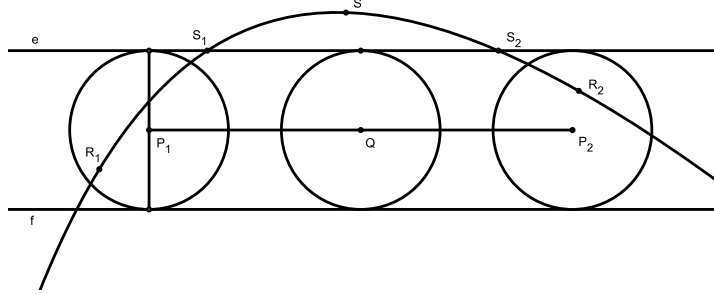


FIGURE 1.

can divide s into the union of continuous arcs s_1 and s_2 intersecting the line e at the points S_1 and S_2 , respectively. The segment between S_1 and S_2 is parallel to the horizontal direction and, at the same time, it intersects the closed disk centered at Q with radius ε . Since L_n^ε is hv-convex the disk and L_n^ε has a common point which is a contradiction. \square

Lemma 8. *Let $\varepsilon > 0$ be an arbitrary positive real number and consider a finite simple polygonal chain \mathcal{P} in the plane. Then*

$$A(\mathcal{P}^\varepsilon) \leq 2\varepsilon \cdot \text{lenght } \mathcal{P} + \varepsilon^2 \pi$$

where $\text{lenght } \mathcal{P}$ is the lenght of \mathcal{P} .

Proof. The proof is an induction for the number n of line segments constituting \mathcal{P} . If $n = 1$, i.e. the polygonal chain consists of only one segment the statement is obviously true (especially we have equality). Suppose that the statement is true for all polygonal chain consisting of n segments and consider a chain \mathcal{P}_{n+1} with vertexes $P_1, \dots, P_{n+1}, P_{n+2}$. Taking

$$\mathcal{P}_{n+1} = \mathcal{P}_n \cup [P_{n+1}, P_{n+2}]$$

we have $(\mathcal{P}_{n+1})^\varepsilon$ as the union of $(\mathcal{P}_n)^\varepsilon$, a rectangle and the disk centered at P_{n+2} with radius ε .

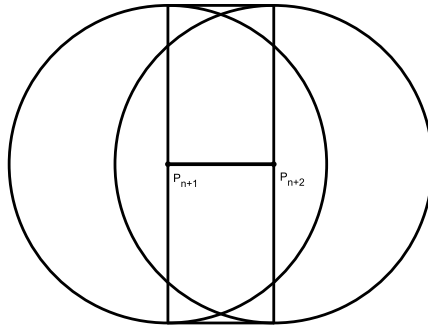


FIGURE 2.

The union of the rectangle and the disk centered at P_{n+1} with radius ε covers the half of the disk centered at P_{n+2} with the same radius and vice versa. Therefore

$$A((\mathcal{P}_{n+1})^\varepsilon) \leq A((\mathcal{P}_n)^\varepsilon) + 2\varepsilon \cdot d_2(P_{n+1}, P_{n+2}) + \varepsilon^2\pi - 2 \cdot \frac{\varepsilon^2\pi}{2},$$

i.e.

$$A((\mathcal{P}_{n+1})^\varepsilon) \leq A((\mathcal{P}_n)^\varepsilon) + 2\varepsilon \cdot d_2(P_{n+1}, P_{n+2})$$

and the inductive hypothesis gives the estimation

$$A(\mathcal{P}_{n+1}^\varepsilon) \leq 2\varepsilon \cdot \text{lenght } \mathcal{P}_{n+1} + \varepsilon^2\pi$$

as was to be stated. \square

Theorem 3. *Suppose that $L \subset \mathbb{R}^2$ is a non-empty bounded closed (i.e. compact) and hv-convex set with a non-degenerate box B . Then*

$$A(L^\varepsilon) - A(L) \leq 2k\varepsilon + \varepsilon^2\pi,$$

where k denotes the perimeter of B .

Proof. Let $r_n \rightarrow 0^+$ be an arbitrary sequence, $B := [a, b] \times [c, d]$ and consider partitions $x_0 = a < x_1 < \dots < x_m = b$, $y_0 = c < y_1 < \dots < y_m = d$ such that $\text{diam } B_{ij}^n < r_n$, where

$$B_{ij}^n := [x_{i-1}, x_i] \times [y_{j-1}, y_j] \quad (i, j = 1, 2, \dots, m).$$

Taking a set $L \in C^+$ let us define L_n as the union of B_{ij}^n 's with $B_{ij}^n \cap L \neq \emptyset$. It is clear that $B(L) = B(L_n) = B$ and L_n is compact. On the other hand $L \subset L_n$ and $H(L_n, L) \leq r_n$ which means that L_n tends to L with respect to the Hausdorff metric. Since L_n is an outer Hausdorff approximation of L it is regular in the sense that $A(L_n) \rightarrow A(L)$. Therefore for any positive real number $\varepsilon > 0$

$$A(L^\varepsilon) - A(L) = A(L^\varepsilon) - \lim_{n \rightarrow \infty} A(L_n) = \lim_{n \rightarrow \infty} (A(L^\varepsilon) - A(L_n))$$

and thus

$$A(L^\varepsilon) - A(L) \leq \lim_{n \rightarrow \infty} (A(L^\varepsilon) - A(L_n))$$

using again that $L \subset L_n$. We are going to prove that L_n is hv-convex. The discussion will be restricted to the convexity into the horizontal direction (the discussion of the vertical convexity is similar). Suppose, in contrary, that there exist points $P_1(x_1, y)$, $P_2(x_2, y) \in L_n$ such that the segment joining P_1 with P_2 contains a point $Q(x, y) \notin L_n$. This means that P_1 and P_2 are in disjoint subrectangles $B_{i_1 j_1}^n$ and $B_{i_2 j_2}^n$, respectively. According to the definition of L_n these subrectangles contain points R_1 and $R_2 \in L$ but the subrectangle $B_{i_3 j_3}^n$ containing Q must be disjoint from the set L . Since L is compact we can choose a positive real number $0 < \delta$ such that $L^\delta \cap B_{i_3 j_3}^n = \emptyset$. Lemma 6 implies that L^δ is connected together with its interior. Therefore $\text{int } L^\delta$ is arcwise connected as a connected open subset of the euclidean plane and the points R_1, R_2 can be joined by a continuous arc in the interior of L^δ . Then the argumentation can be finished in the same way as in Lemma 7 (see figure 1 with squares instead of circles). The hv-convexity of L_n implies its connectedness too because of the same reason as in the proof of lemma 6: it can be considered as a special kind of parallel

body constructed from L by adding rectangles instead of circles. Consider now the boundary of L_n as a finite simple polygonal chain \mathcal{P}_n in the plane. Length \mathcal{P}_n is just the perimeter of the box B because of the hv-convexity of L_n . Since $L_n^\varepsilon \setminus L_n \subset \mathcal{P}_n^\varepsilon$ we have by Lemma 8 that

$$A(L_n^\varepsilon) - A(L_n) = A(L_n^\varepsilon \setminus L_n) \leq A(\mathcal{P}_n^\varepsilon) \leq 2k\varepsilon + \varepsilon^2\pi$$

and, consequently,

$$A(L^\varepsilon) - A(L) \leq 2k\varepsilon + \varepsilon^2\pi$$

as was to be stated. \square

Theorem 4. *The function $\Phi: L \in \mathcal{C}^+ \subset \mathcal{F}(\mathbb{R}^2) \rightarrow \Phi(L) := f_L$ is continuous.*

Proof. According to Lemma 1 it is enough to prove that for any sequence $L_n \rightarrow L$

$$f_L(x, y) \leq \liminf_{n \rightarrow \infty} f_{L_n}(x, y) \quad ((x, y) \in \mathbb{R}^2).$$

Since Φ preserves the ordering with respect to the inclusion we have that

$$f_L(x, y) \leq f_{(L_n)^{r_n}}(x, y) = f_{L_n}(x, y) + \int_{(L_n)^{r_n} \setminus L_n} (\alpha, \beta) \mapsto d_1((x, y), (\alpha, \beta)),$$

where $r_n := H(L_n, L)$. Therefore

$$f_L(x, y) \leq f_{L_n}(x, y) + N(x, y, r_n)(A((L_n)^{r_n}) - A(L_n)),$$

where

$$N(x, y, \varepsilon) := \max\{d_1((x, y), (\alpha, \beta)) \mid (\alpha, \beta) \in L^{2\varepsilon}\}$$

because the inclusion $L_n \subset L^{r_n}$ shows that $(L_n)^{r_n} \subset L^{2r_n}$. Suppose that $N(x, y, \varepsilon)$ is attained at the point $(\alpha^\varepsilon, \beta^\varepsilon) \in L^{2\varepsilon}$. According to the definition of the outer parallel body, there exists $(\alpha, \beta) \in L$ such that

$$d_2((\alpha, \beta), (\alpha^\varepsilon, \beta^\varepsilon)) \leq 2\varepsilon$$

and thus

$$\begin{aligned} d_1((x, y), (\alpha^\varepsilon, \beta^\varepsilon)) &\leq d_1((x, y), (\alpha, \beta)) + d_1((\alpha, \beta), (\alpha^\varepsilon, \beta^\varepsilon)) \leq \\ &\leq N(x, y) + 2\sqrt{2}\varepsilon, \quad \text{where } N(x, y) := N(x, y, 0). \end{aligned}$$

Therefore

$$(9) \quad f_L(x, y) \leq f_{L_n}(x, y) + (N(x, y) + 2\sqrt{2}r_n)(A((L_n)^{r_n}) - A(L_n)).$$

Using Theorem 3

$$A((L_n)^{r_n}) - A(L_n) \leq 2kr_n + (r_n)^2\pi$$

and we have that

$$f_L(x, y) \leq \liminf_{n \rightarrow \infty} f_{L_n}(x, y)$$

as was to be proved. \square

Theorem 5. (A theorem of localization II.) *Suppose that $K \subset \mathbb{R}^2$ is a non-empty bounded closed (i.e. compact) and hv-convex set with a non-degenerate box B . For any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$|f_L - f_K| := \sup_{(x, y) \in B} |f_L - f_K|(x, y) < \delta \quad (L \in \mathcal{C}^+)$$

implies that $L \in (\Phi^{-1}(f_K)^\varepsilon \cap \mathcal{C}^+$. Therefore if $f_L \approx f_K$ then L approximates at least one of the sets in \mathcal{C}^+ with the same coordinate X-rays as K almost everywhere.

Proof. Let $\varepsilon > 0$ be given and suppose, in contrary, that there exists a sequence $(L_n)_{n \in \mathbb{N}}$ in \mathcal{C}^+ such that $|f_{L_n} - f_K| < \delta_n \rightarrow 0^+$ but $L_n \notin (\Phi^{-1}(f_K)^\varepsilon \cap \mathcal{C}^+$. After choosing a convergent subsequence $L_{m(n)} \rightarrow L \in \mathcal{C}^+$ we have, by Theorem 4, that $f_{L_{m(n)}} \rightarrow f_K = f_L$ but $L \notin (\Phi^{-1}(f_K)^\varepsilon \cap \mathcal{C}^+$ which is a contradiction. \square

4. AN APPLICATION

Theorem 5 admits an approximating process via GC functions to find hv-convex bodies with given coordinate X-rays. In what follows we formulate the steps of the basic algorithm. The difficulty is that the coordinate X-rays do not determine the bodies even if they are convex. Gardner and McMullen [2] proved that there are four directions in the plane such that the X-rays of any convex planar body in these direction determine the body uniquely among all planar convex bodies. As a recent result Gardner and Kiderlen [3] present new algorithms for reconstructing planar convex bodies from their parallel X-rays in situations that guarantee a unique solution. Here we use only the coordinate X-rays in terms of the corresponding GC function of a compact hv-convex planar set K . Let $[a, b]$ and $[c, d]$ be the sides of the box and choose a positive number $\varepsilon > 0$. We sketch the steps of an algorithm how to find $L \in (\Phi^{-1}(f_K)^\varepsilon \cap \mathcal{C}^+$ (see Theorem 5).

After reconstructing the generalized conic function f_K consider the partitions $x_0 = a < x_1 < \dots < x_n = b$, $y_0 = c < y_1 < \dots < y_n = d$ of the intervals $[a, b]$ and $[c, d]$ into n equal parts. Let

$$B_{ij}^n := [x_{i-1}, x_i] \times [y_{j-1}, y_j],$$

where $(i, j = 1, 2, \dots, n)$. The sequence of the set $H_0 := B$,

$$H_n := \bigcup \left\{ B_{ij}^n \mid B_{ij}^n \cap K \neq \emptyset, \text{ where } i, j = 1, 2, \dots, n \right\}$$

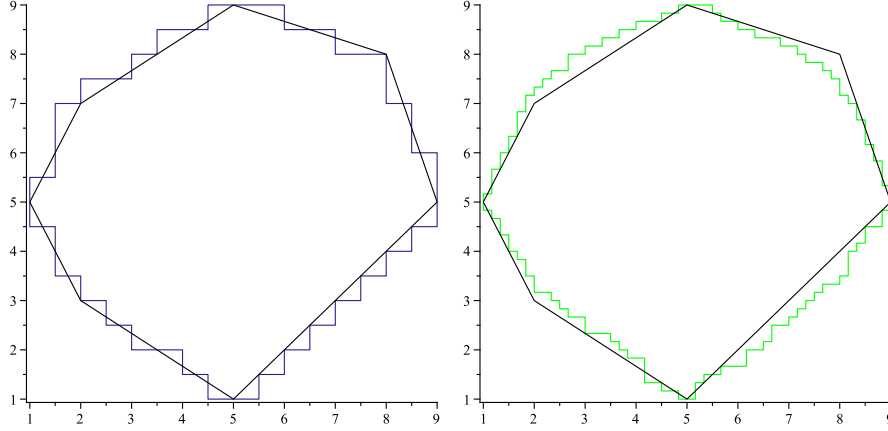
tends to K in \mathcal{C}^+ with respect to the Hausdorff metric and for all $n \in \mathbb{N}$ $K \subset H_n$. Therefore the generalized conic functions f_{H_n} tends to f_K .

After choosing an $n \in \mathbb{N}$ we have finitely many sets in \mathcal{C}^+ which are the union of some of the sets B_{ij}^n 's. Let L_n be one of them minimizing the difference

$$|f_{L_n} - f_K| := \sup_{(x,y) \in B} |f_{L_n} - f_K|(x, y).$$

Since f_{H_n} tends to f_K , so does f_{L_n} . By Theorem 5 there exists $\delta > 0$ such that $|f_{L_n} - f_K| < \delta$ implies that $L \in (\Phi^{-1}(f_K)^\varepsilon \cap \mathcal{C}^+$. If N is great enough to satisfy $|f_K - f_{L_N}| < \delta$ then we put $L := L_N$.

The number of unions grows fast as n increases. This problem can be handled by using greedy algorithm in the following way. Starting from B (as the union of all subrectangles) we decrease the difference $|f_K - f_B|$ by deleting one of the subrectangle from B , i.e. $B_1 := \bigcup_{i \neq i_1, j \neq j_1} B_{ij}^n$ is a set in \mathcal{C}^+ and $|f_K - f_{B_1}| \leq |f_K - f_B|$. If there is no such a set then we increase the value of n and repeat the procedure again. Otherwise we choose one of the

FIGURE 3. The greedy algorithm in case of $n = 16$ and $n = 48$.

subrectangles causing the largest decrease and repeat the first step with B_1 instead of B . This procedure does not guarantee the optimal solution L_n but computable and quick. Figure 3 shows results with greedy algorithm in case of $n = 16$ and $n = 48$.

5. THE CASE OF COMPACT CONVEX PLANAR BODIES: GARDNER'S PROBLEM

Let $K \subset \mathbb{R}^2$ be a non-empty bounded closed (i.e. compact) and convex set with a non-degenerate box B and consider the collection \mathcal{C}^{++} of non-empty bounded closed (i.e. compact) convex sets having the same box as K . Radström theorem [4] states that the space of non-empty compact convex subsets in \mathbb{R}^2 (with the Hausdorff metric) can be embedded isometrically into a linear normed space. Lattice properties [1] were added by A. G. Pinsker in 1966 proving that there also exists an isometric order preserving embedding into a Kakutani vector space V . If W is the Kakutani vector space of convex function $f: B \rightarrow \mathbb{R}$ equipped with the supremum norm and partial ordering

$$|f| := \sup_{(x,y) \in B} |f(x,y)| \text{ and } f \geq g$$

in the usual sense $f(x,y) \geq g(x,y)$ for all $(x,y) \in B$, respectively then the correspondence of GC functions to the bodies can be interpreted as a bounded order-preserving concave and continuous mapping $\Phi: \mathcal{C}^{++} \subset V \rightarrow W$, where \mathcal{C}^{++} is a convex compact subset, V and W are Kakutani vector spaces (see Lemma 4, Theorem 1 and Theorem 4).

Corollary 3. *If K is in the relative interior of \mathcal{C}^{++} then the one-sided directional derivatives*

$$\Phi'(K)(K, M)(x, y) := \lim_{t \rightarrow 0^+} \frac{\Phi((1-t)K + tM) - \Phi(K)}{t}(x, y)$$

exists for any direction represented by the pair (K, M) , where $M \in \mathcal{C}^{++}$ and $(x, y) \in B$

Since Φ is a continuous mapping defined on a compact metric space its inverse (as a set-valued function) is upper semi-continuous, i.e. for any $L \in \mathcal{C}^{++}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $|f_M - f_L| < \delta$ implies that $M \in (\Phi^{-1}(f_L))^\varepsilon \cap \mathcal{C}^{++}$ (cf. Theorem 5). The mapping Φ^{-1} is *lower semi-continuous* at $f_L \in \Phi(\mathcal{C}^{++})$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f_M - f_L| < \delta$ implies that $L \in (\Phi^{-1}(f_M))^\varepsilon \cap \mathcal{C}^{++}$. In what follows we prove that the lower semicontinuity is related to the unicity of the body with given coordinate X-rays. To *characterize those convex bodies that can be determined by two X-rays* is an open problem due to R. J. Gardner [2], Problem 1.1, p. 51. According to the affine nature of the problem we can suppose that the X-ray directions correspond to the coordinate axes without loss of generality.

Theorem 6. *The set $K \in \mathcal{C}^{++}$ is determined by the coordinate X-rays if and only if the mapping Φ^{-1} is lower semi-continuous (esp. continuous) at f_K .*

Proof. Consider a sequence $f_{L_n} \rightarrow f_K$ and suppose that

$$\limsup_{n \rightarrow \infty} \text{diam } \Phi^{-1}(f_{L_n}) = \varepsilon_* > 0,$$

where $\text{diam } \Phi^{-1}(f_{L_n})$ is the diameter of $\Phi^{-1}(f_{L_n}) \subset \mathcal{C}^{++}$ with respect to the Hausdorff distance. By the upper semi-continuity of Φ^{-1}

$$\text{diam } \Phi^{-1}(f_K) \geq \limsup_{n \rightarrow \infty} \text{diam } \Phi^{-1}(f_{L_n}) > 0$$

and K cannot be determined by the coordinate X-rays. Therefore if K is determined by the coordinate X-rays then

$$\limsup_{n \rightarrow \infty} \text{diam } \Phi^{-1}(f_{L_n}) = 0 \Rightarrow \liminf_{n \rightarrow \infty} \text{diam } \Phi^{-1}(f_{L_n}) = 0$$

and $\Phi^{-1}(f_{L_n})$ tends to a singleton in $\Phi^{-1}(f_K) = \{K\}$ because of the upper semi-continuity. Conversely suppose that the inverse mapping is lower semi-continuous (esp. continuous) at f_K . Since the sets that can be determined by their coordinate X-rays form a dense subset [2] in \mathcal{C}^{++} we can choose a sequence $f_{L_n} \rightarrow f_K$ such that $\Phi^{-1}(f_{L_n})$ is a singleton, so is $\Phi^{-1}(f_K)$ because of the continuity at f_K . \square

Remark 6. Conditions in terms of K providing the lower semi-continuity of Φ^{-1} at f_K are solutions of Gardner's problem.

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